Tetris as an introduction to Krohn-Rhodes and semigroup theory

Peter C. Jentsch, Chrystopher Nehaniv

University of Waterloo

February 18, 2021

Tetris

- Tetris is an arcade puzzle game created by Alexey Pajitnov in 1984, that has since become a worldwide cultural phenomenon, and one of the most popular video games of all time.
- Players attempt to stack polyominoes as efficiently as possible

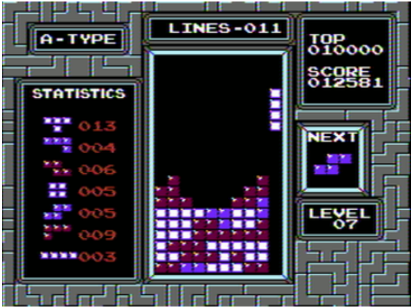

Figure: Tetris gameplay screenshot

By Nestopia screenshot,

https://en.wikipedia.org/w/index.php?curid=6088342

Mathematical results on Tetris

- There exists a finite sequence of pieces the computer can play which are unplaceable [Burgiel, 1997]
- Solving most of the relevant problems are NP-complete, and difficult to approximate [Demaine et al., 2003]
- Almost every possible arrangement of blocks in the Tetris board is constructable from the Tetris pieces, under Tetris rules [Hoogeboom and Kosters, 2005]

Semigroup Theory 1

Some relevant facts and definitions needed

- A transformation semigroup is a semigroup consisting of functions from a set to itself (transformations of the set), analogous to permutation groups
- If S contains the identity, it is a transformation monoid. S^I denotes S coupled with an identity transformation, so S^I is always a monoid.
- Let S, H be finite semigroups, acting on the sets X, Y respectively. We write their respective transformation semigroups as (X, S) and (Y, H).

Semigroup Theory 2

- We say (X, S) divides (Y, H) if S is homomorphic to a subsemigroup of H
- (X, S) embeds in (Y, H) is S is isomorphic to a subsemigroup of H
- The *wreath product* of (X, S) with (Y, H) is written $(X, S) \wr (Y, H) = (X \times Y, W)$. W is the set of all $w = (f, h)$, where $f : Y \to S$ and $h \in H$. W acts on $X \times Y$, such that, for all $(x, y) \in X \times Y$,

$$(x, y) \cdot w = (x \cdot f(y), y \cdot h)$$

.

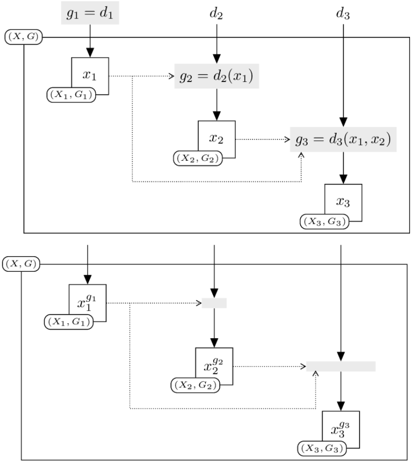

Figure: Cascade product of $(X_1, G_1), (X_2, G_2), (X_3, G_3)$ [Egri-Nagy et al., 2014]

Tetris as a transformation semigroup

- Let P a set of pieces. A "piece" is a group of connected cells.
- The game is played on an $n \times k$ sized board
- Define S as the semigroup where each $\sigma = (p, \xi) \in S$ consists of a set of connected cells $p \in P$, and a position $1 \leq \xi \leq n$ (although the precise limits on the position ξ depend on the width of p).
- An element $\sigma \in S$ acts on a configuration x by "dropping" the piece p with the *leftmost* block in the column x, and if there is a full row of width n, it is removed, and the blocks above the row are dropped down by one.
- If the height of the stack exceeds k cells, then $x \cdot \sigma = E$. Furthermore, $E \cdot \sigma = E$ for all $\sigma \in S$.

Tetris as a transformation semigroup

For $\sigma_1, \sigma_2 \in S$, define their product $\sigma_1\sigma_2$ as the transformation resulting from applying σ_1 then σ_2 in the above way.
Let X be the set of configurations reachable by applying any word in S to the empty configuration e.

Definition

(X, S) is a finite transformation semigroup, which we will call the **Tetris** semigroup of P, a set of polyominoes, on the board with dimensions $n \times k$.

- Pieces in the standard game are the set of tetrominoes
- Generalizations with simpler sets, such as the triominoes, will be discussed here

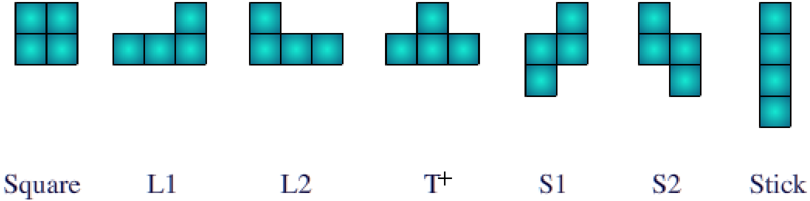

Figure: Standard Tetris pieces with labels

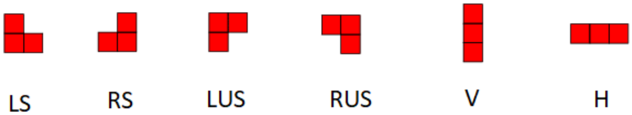

Figure: Triominoes with labels

Krohn-Rhodes Theory

Theorem (Krohn-Rhodes decomposition)

Given a transformation semigroup, S, the KR decomposition of S is $H_1, H_2, ...H_n$, such that

$$S \text{ divides } H_1 \wr H_2 \wr H_3 ... \wr H_n$$

where the H_i are finite simple groups, or the flip-flop monoid.
[Krohn and Rhodes, 1965]

The Krohn-Rhodes (KR) theorem describes a general decomposition of transformation semigroups in terms of wreath products of the finite simple groups and the flip-flop monoid.

Flip-flop and Identity-reset semigroups

The flip-flop monoid is a three element semigroup $S = \{A, B, I\}$ acting on $X = \{1, 2\}$, such that I is an identity and $\forall x \in X$,

$$x \cdot A = 1$$

$$x \cdot B = 2$$

Important as the only non-reversible element in the KR decomposition.
Also called an identity-reset monoid on two elements.

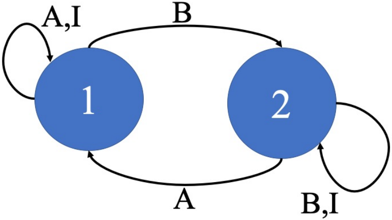

Figure: Flip-flop Monoid

Holonomy groups

- Although the proof of the Krohn-Rhodes theorem gives a method for computationally obtaining the decomposition of a given semigroup, the decomposition obtained is often very far from optimal.
- Holonomy groups give us the Holonomy decomposition theorem, which implies the KR theorem, and gives a more efficient decomposition. [Eilenberg, 1974]

Holonomy Groups: tiles

- Define
$$Q = \{\{X \cdot s\} | s \in S\} \cup \{X\} \cup \{\{a\} | a \in X\}$$
.
- Given, $A, B \in Q$, we define an reflexive, transitive relation on Q,
$$A \leq B \iff \exists s \in S^I, A \subseteq B \cdot s$$
Furthermore, let $A < B$ if $A \leq B$ but not $B \leq A$.
- This relation, which we will call *subduction* gives us an equivalence relation on Q: $A \equiv B \iff A \leq B, B \leq A$
- Define A to be a *tile* of B if $A \subsetneq B$ and
$$\forall Z \in Q, A \leq Z \leq B \implies Z = A, Z = B$$
- For each equivalence class $A \backslash \equiv$ in $Q \backslash \equiv$, let $\bar{A}$ be the unique representative.

Holonomy Groups

- If $A \in Q$, the set of tiles of A is $\Theta_A \subset Q$
- The *holonomy group*, written H_A, of A is the set of permutations of θ_A induced by the elements of S^I
- If we let H_A act on Θ_A, then (Θ_A, H_A) is the holonomy permutation group of A
- We can define the height of $A \in Q$ by $h(A)$, where $h(A)$ is the length of the longest strict subduction chain.
- Let $h = h(X)$ be the height of (X, S). For each $i \in [1, h]$, let

$$(\Phi_i, \mathfrak{H}_\mathfrak{i}) = \prod_{i=1, h(\bar{A})=i}^{h} (\Theta_{\bar{A}}, H_{\bar{A}})$$

- $(\Phi_i, \mathfrak{H}_\mathfrak{i})$ is a permutation group and $(\Phi_i, \bar{\mathfrak{H}}_\mathfrak{i})$ is the permutation-reset transformation obtained by appending all constant maps to $\mathfrak{H}_\mathfrak{i}$.

Holonomy Decomposition Theorem

Theorem (Holonomy decomposition theorem)

Let (X, S) be a finite transformation semigroup, then

$$(X, S) \text{ divides } (\Phi_1, \bar{\mathfrak{H}}_1) \wr (\Phi_2, \bar{\mathfrak{H}}_2) \wr \ldots \wr (\Phi_h, \bar{\mathfrak{H}}_{\mathfrak{h}})$$

[Eilenberg, 1974]
This theorem implies the KR decomposition theorem from before.
We will use the Holonomy implementation in SGPDec, by A. Egri-nagy, for the analysis. [Egri-Nagy et al., 2014]

Analysis - Standard rules

- Standard tetris, with tetrominoes on a 10 x 20 board has an extrememly large state space (on the order of 2^{200})
- Our analysis will be exploring the semigroup via Tri-tris on small board sizes.
- Under the standard rules, tetris seems to be aperiodic, since it is always possible to "escape" a permutation group. Therefore, the KR decomposition consists entirely of flip-flop monoids.
- In the following section we will modify the rules to introduce periodicity and therefore, groups in the decomposition.

Aperiodic Complexity and Tetris

- KR complexity of a semigroup is the smallest number of finite simple groups in its KR decomposition
- Since aperiodic semigroups have zero finite simple groups in their KR decomp., we need another metric. Therefore, we use the smallest number of direct products of flip-flop monoids in the KR decomposition.

For very small game sizes, we can use GAP and SGPDec to obtain the complexity. We can bound the aperiodic complexity from above by the length of the longest strict subduction chain of (X, S), called the *height* of X with respect to S.
We define the height of the semigroup (X, S) as

$$h_s(X) = |\{X_1 \subset_S X_2 \subset_S X_3 ... X\}| - 1$$

For some small values of n, k these bounds are shown below.

Board Dimensions	No. of configurations, $\lvert X\rvert$	$\lvert S\rvert$	$h_s(X)$
3×3	35	2056	13
3×4	135	259726	32

A rule modification

- We remove the state E, and replace it with the empty board. Therefore, any losing game "resets" to a new game.
- This rule modification introduces a lot more symmetry into the semigroup, causing some interesting groups to appear in the holonomy decomposition.

Board Dimensions	$\|X\|$	holonomy groups present
3×3	34	$(4, C_2 \times C_2), (3, S_3), (2, C_2)$
$3 \times 4,\ P = \{\mathrm{RS}, \mathrm{LUS}, \mathrm{RUS}\}$	116	$(4, C_2), (5, S_5), (4, S_4), (3, S_3), (2, C_2)$

Holonomy Group $(4, C_2 \times C_2)$

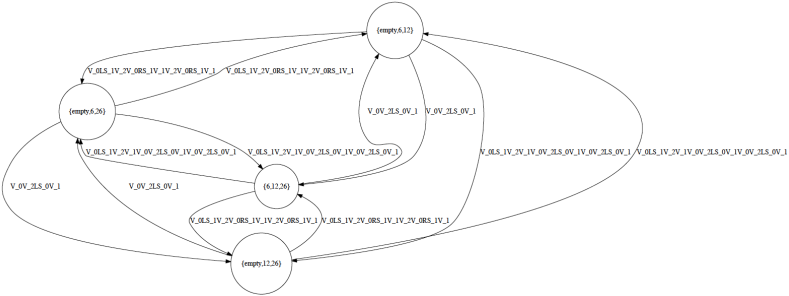

Figure: Graph of the Holonomy group showing tiles as nodes, and members of S as edges

Visualization of states

The above group is permuting the following states:

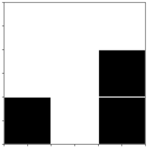

Figure: State 12

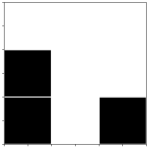

Figure: State 26

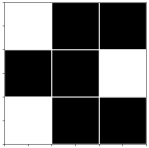

Figure: State 6

Discussion

- We formulated the game of Tetris as a transformation semigroup
- We have seen how KR complexity can be used to analyze a game formulated as a semigroup
- Tetris seems to exhibits pools of reversibility only if we add a "restart" mechanic
- Is Tetris with regular rules provably aperiodic?

References I

Burgiel, H. (1997).
How to lose at tetris.
The Mathematical Gazette, 81(491):194–200.

Demaine, E. D., Hohenberger, S., and Liben-Nowell, D. (2003).
Tetris is hard, even to approximate.
In *International Computing and Combinatorics Conference*, pages 351–363. Springer.

Egri-Nagy, A., Mitchell, J. D., and Nehaniv, C. L. (2014).
Sgpdec: Cascade (de) compositions of finite transformation semigroups and permutation groups.
In *International Congress on Mathematical Software*, pages 75–82. Springer.

Eilenberg, S. (1974).
Automata, languages, and machines.
Academic press.

References II

Hoogeboom, H. J. and Kosters, W. A. (2005).
The theory of tetris.
Nieuwsbrief van de Nederlandse Vereniging voor Theoretische Informatica, 9:14–21.

Krohn, K. and Rhodes, J. (1965).
Algebraic theory of machines. i. prime decomposition theorem for finite semigroups and machines.
Transactions of the American Mathematical Society, 116:450–464.